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LETTER TO THE EDITOR

Breakdown of the soliton-gas phenomenology for the classical statistical mechanics of the sine-Gordon model

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Abstract. It is shown that the soliton-gas phenomenology breaks down for the classical statistical mechanics of the sine-Gordon model. As a main cause of the breakdown, it is pointed out that the solitons and breathers in the gas phenomenology do not precisely represent the total degrees of freedom of the system.

The gas phenomenological approach to the statistical mechanics of soliton-bearing systems, initiated by Krumhansl and Schrieffer [1] and improved and extended by a number of authors [2–5] over the last decade, appears to be nearing its final goal: i.e. describe exactly the statistical-mechanical properties of completely integrable soliton-bearing systems in terms of zero-temperature soliton energies, soliton densities and two-body phaseshifts among solitons. Here the term solitons represents all the exact excitations in a given system. Although in classical mechanics terms the goal has not been justified by anything more than a naive intuition. Its quantum version has already been reached as the Bethe–Ansatz (BA) formulation and more completely as the factorized S-matrix formulation [6] of the quantum statistical mechanics of completely integrable systems such as the nonlinear Schrödinger model [7], the quantum Toda lattice [8, 9] and the massive-Thirring-sine-Gordon model [10].

Recently in the case of the Toda lattice, Theodorakopoulos [11] and Opper [12] have shown that the proper classical limit $\hbar \to 0$ of the quantum BA thermodynamics gives a gas phenomenological formulation which rigorously reproduces the Toda classical free energy [13]. Based on intuition and encouraged by this new evidence, one may quite naturally expect that the gas phenomenological description is exact for any completely integrable systems. In this paper, however, I shall show that the gas phenomenological description breaks down for the classical statistical mechanics of the sine-Gordon (SG) model, particularly at low temperatures.

Currently, for the classical SG thermodynamics, there exist three different BA formulations due to (A) Chen *et al* [14], (B) Chung [15] and (C) Timonen *et al* [16], and two different soliton-gas phenomenologies due to (D) Sasaki [5] and (E) Takayama and Ishikawa [4] although (A) and (C) turn out to be equivalent and (D) and (E) turn out to be equivalent. The BA formulation (A) = (C) is written in terms of solitons and phonons and therefore may be referred to as the soliton–phonon theory. On the other hand, the BA formulation (B) and the gas phenomenology (D) = (E) are both written in terms of solitons and breathers and may be referred to as the soliton–breather theories.

In this paper, we will focus on the soliton–breather theories. The elucidation of the possible exactness of the soliton–phonon theory is the last leftover question in the long-standing breather-phonon problem [17] and requires careful considerations. My preliminary study shows that the soliton-phonon theory is no more than a good approximation in the weak coupling regime [18]. This leads to a speculation that the nonlinear phonon is not an exact notion in the SG model, suggesting the re-examination of the classical inverse scattering analysis [19] as the only source of reporting the nonlinear phonon as an exact excitation.

We now examine the soliton-breather theories in detail. First of all, like all the previous classical thermodynamic theories except those of Theodorakopoulos and Opper, the factor $2\pi\hbar$ arising from quantization of particle momenta, still remains in the gas phenomenologies (D) = (E). Nevertheless, with a proper treatment of the factor $2\pi\hbar$ as given by Opper, one can demonstrate that these formulations give the gas phenomenological equation which correctly describes the classical Toda thermodynamics. When applied to the SG model, these formulations, particularly the Sasaki gas phenomenology (cf (29)-(31) in [5]), after a proper treatment of the factor $2\pi\hbar$, are found to provide the identical integral equations for the SG thermodynamics as in the BA formulation (B) (explicit forms are given in equations (7), (15) and (16) in [15] but are not necessary here). The point is that the basic equations obtained for the thermodynamics take the form of the gas phenomenology. That is, they are written solely in terms of the zero-temperature energies of solitons and breathers, their concentrations and the classical two-body phase shifts among solitons and breathers.

Based on the basic equations, Sasaki [5] and Chung [15] reproduced, analytically and numerically, the transfer matrix method result [20]. In particular, Sasaki has shown that the total number of breathers is half the total number of degrees of freedom of the system, which is consistent with the fact that a breather has an internal degree of freedom as well as a translational degree of freedom. It is noted, however, that the two soliton-breather theories both limit their arguments essentially to high temperatures. In the gas phenomenology (D), temperature is restricted to $m \ll T$, where m is the phonon mass $\sim\hbar$. At first sight, this condition is always met because $\hbar\to 0$ in the classical limit. However, as noted above, the factor $2\pi\hbar$ should be chosen to show the theory to be a genuine classical theory, and I find that the proper classical limit $\hbar\to 0$ effectively puts $m=\frac{1}{2}\pi$. The condition $m \ll T$ then implies a high temperature. Similarly in the BA formulation (B), the restriction $Tl > 1$ applied, where l is a lattice cut-off. Since l^{-1} is of the order of the maximum momentum of unit-mass particles, which in the relativistic model is approximately the maximum energy of unit-mass particles, and since the unit-mass here is literally unity (the notation here is such that the classical soliton mass $= g_0/2$ and the weak coupling limit is $g_0^{-1}\to 0$) and not the phonon mass of order $\hbar$, the above condition also implies a high temperature. Therefore, the validity of the gas phenomenological description at low temperatures is yet to be clarified.

Let us now show that the gas phenomenological description of the SG thermodynamics breaks down at low temperatures. This is clearly seen in the weak coupling limit, where solitons and antisolitons disappear and the basic equations for the thermally renormalized breather energy $\mathscr{E}(\theta, \alpha)$ and the free energy per unit length F/L give [15]

$$\mathscr{E}(\theta, \alpha) = \theta \cosh \alpha + 8\pi^2 T \int_0^\infty \mathrm{d}\theta' \min(\theta, \theta') \exp[-\mathscr{E}(\theta', \alpha)/T] \tag{1}$$

$$F/L = -2\pi T \int_{-\infty}^{\infty} \mathrm{d}\alpha \cosh \alpha \int_0^\infty \mathrm{d}\theta\, \theta \exp[-\mathscr{E}(\theta, \alpha)/T]. \tag{2}$$

The integral equation (1) can be solved easily to give

$$\mathscr{E}(\theta, \alpha) = -2T \ln\left(\frac{x}{2\pi \sinh(x\theta + A)}\right) \tag{3}$$

where

$$x \equiv (\cosh \alpha)/2T \qquad A \equiv \sinh^{-1}(x/2\pi). \tag{4}$$

Substituting (3) and (4) into (2) gives

$$F/L = \frac{T}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\alpha \cosh \alpha \ln(x/\pi) - \frac{T}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\alpha \cosh \alpha\, A \tag{5}$$

where the first term is the free energy of free phonons. At high temperatures, the second term becomes unimportant relative to the first term, agreeing with the previous analysis. At low temperatures, however, $A \sim \ln(x/\pi)$ and we have, to leading order in T,

$$F/L \sim -2\pi^2 T^3 \tag{6}$$

which is far from correct for the free energy of free phonons. Furthermore, even at high temperatures, large values of α such that $\cosh \alpha \gg T$ does not contribute to the free energy because $A \sim \ln(x/\pi)$, lead to a finite free energy free from an ultraviolet divergence, which is simply wrong.

We thus conclude that the gas phenomenological description of the SG thermodynamics is not exact but is approximately correct only at high temperatures and with an appropriate lattice cut-off.

The gas phenomenology works perfectly for the classical Toda lattice. What is wrong with it for the classical SG thermodynamics? An essential difference between the two models is in the chemical potentials. In the Toda lattice, which has a similar physical structure as the nonlinear Schrödinger model, the chemical potential for the soliton is finite and the number of solitons is precisely controlled to be the same as the number of degrees of freedom of the system. On the other hand, in the SG model the chemical potentials for breathers and solitons are all zero with in the charge neutral sector which is our current subject, and the number of breathers and solitons is not obviously related to the degrees of freedom of the system. Nevertheless, the existing soliton-breather theories implicitly assume as a trivial fact that the total degrees of freedom of the system are precisely represented by the soliton-breather gas. Positive evidence for this assumption was provided by Sasaki as he consistently demonstrated that twice (translation and internal) the total number of breathers is equal to the total degrees of freedom of the system when the soliton contribution is negligible. In the rest of this paper, however, I shall demonstrate that the gas phenomenological description is not self-consistent: that is, its result contradicts its starting assumption in the above. As for the Sasaki proof, it will be shown to be true only at high temperatures and with an appropriate lattice cut-off.

We start with the momentum conservation of the ith breather in the quantum BA formulation

$$\frac{\mathrm{d}P_j(\alpha)}{\mathrm{d}t} = 2\pi\hbar(\rho_j + \tilde{\rho}_j) + \sum_i \dot{\Delta}_{ji} * \rho_i \tag{7}$$

where ρ_j and $\tilde{\rho}_j$ represent densities of breather and missing breather, $*$ denotes a convolution with respect to rapidity α, and $\dot{\Delta}_{ji}$ is the α derivative of the quantum two-body phaseshift. Since we will finally take the weak coupling limit $g_0^{-1} \to 0$, we neglect solitons here. Proceeding as in the BA formulation (B) and defining the classical breather density $\rho(\theta, \alpha)$ by

$$\rho_j \equiv \frac{\hbar}{2g_0} \rho(\theta, \alpha/2) \tag{8}$$

the classical limit $\hbar \to 0$ of (7) gives (see equation (7*C*) of [15] for $\dot{\Delta}_{\theta\theta'}$)

$$g_0 \sin\theta \cosh\alpha = \frac{1}{2\pi g_0} \exp(\mathscr{E}(\theta, \alpha)/T)\rho(\theta, \alpha) + \int_0^{\pi/2} \mathrm{d}\theta' \, \dot{\Delta}_{\theta\theta'} * \rho(\theta'). \tag{9}$$

Now in the weak coupling limit $g_0 \to \infty$, we rewrite $g_0\theta$ as θ and $\rho(\theta/g_0, \alpha)/g_0$ as $\rho(\theta, \alpha)$; thus (9) reduces to

$$\theta \cosh\alpha = \frac{2\pi \sinh^2(x\theta + A)}{x^2} \rho(\theta, \alpha) + 4\pi \int_0^{\infty} \mathrm{d}\theta' \min(\theta, \theta')\rho(\theta'). \tag{10}$$

The total number of breathers in the weak coupling limit is given by

$$N = \int_0^{\infty} \mathrm{d}\theta \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \rho(\theta, \alpha). \tag{11}$$

The integral equation (10) can be solved as follows. First, differentiating twice with respect to θ and defining $\bar{\theta} \equiv x\theta + A$ gives

$$4 \sinh\bar{\theta}\rho + 4\cosh\bar{\theta}\rho' + \sinh\bar{\theta}\rho'' = 0. \tag{12}$$

Now put

$$\rho = \exp\left(\int z \, \mathrm{d}\bar{\theta}\right)(\sinh^2\bar{\theta})^{-1}. \tag{13}$$

Then (12) becomes a Riccati differential equation for z

$$z' + z^2 = 2/\sinh^2\bar{\theta} \tag{14}$$

having the solution

$$z = -\frac{1}{\sinh\bar{\theta}\cosh\bar{\theta}} + \frac{\tanh^2\bar{\theta}}{\bar{\theta} - \tanh\bar{\theta} + C} \tag{15}$$

where C is a constant of integration. Substituting (15) into (13) gives (D is another constant of integration)

$$f = \frac{D}{\sinh^2\bar{\theta}} \frac{\bar{\theta} - \tanh\bar{\theta} + C}{\tanh\bar{\theta}}. \tag{16}$$

By checking the limiting values of ρ at $\theta = 0$ and ∞ from (10), the constants C and D are determined:

$$C = \tanh A - A \qquad D = \frac{x}{2\pi} \cosh\alpha. \tag{17}$$

Substituting (16) and (17) into (11) and performing θ integration gives the total number of breathers (cf (4) for A)

$$N = \frac{1}{4\pi} \int_{-\infty}^{\infty} \mathrm{d}\alpha \cosh\alpha(1 - \tanh A). \tag{18}$$

To see the consistency of the above procedure for the breather density with the gas phenomenological description leading to the free energy (5), one may calculate the internal energy per unit length

$$E/L=\int_{-\infty}^{\infty}\mathrm{d}\alpha\int_{0}^{\infty}\mathrm{d}\theta\,\theta\cosh\alpha\rho(\theta,\alpha)$$
$$=\frac{T}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\cosh\alpha(1-\tanh A) \qquad (19)$$

and check that (5) and (19) do not violate the thermodynamic relationship

$$E=-T^2\frac{\partial}{\partial T}(F/T). \qquad (20)$$

one can easily see that the relationship (20) is indeed satisfied.

As is clear from (18), due to the second term in the bracket, twice the total number of breathers in the weak coupling limit, $2N$, does not agree with the total degrees of freedom of the system

$$N_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\alpha\cosh\alpha. \qquad (21)$$

The undesirable term becomes relatively unimportant at high temperatures and with an appropriate cut-off $\infty\rightarrow\alpha_{\max}$ such that $\cosh\alpha_{\max}\ll 4\pi T$ (note that $\sinh\alpha_{\max}=\pi/l$). Without the cut-off, Sasaki's previous demonstration is correct only at $T=\infty$.

To summarize, I have shown that the gas phenomenological description of the classical sine-Gordon thermodynamics breaks down at low temperatures even with a lattice cut-off. As a main cause of the breakdown, I have pointed out that the solitons and breathers in the gas phenomenology do not precisely represent the total degrees of freedom of the system. Such a result, however, does not imply the inappropriateness of the soliton–breather approach itself. There is no doubt that solitons and breathers are the only necessary ingredients for constructing the classical sine-Gordon thermodynamics. The result only points to a failure of the simple and reasonably looking gas phenomenological picture in that the classical sine-Gordon thermodynamics can be described solely in terms of zero-temperature energies of solitons and breathers, their concentrations, and the classical two-body phase shifts among solitons and breathers. The exact theory of the classical statistical mechanics of the sine-Gordon model is still an open question.

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